AN ANALYTIC STUDY OF THE PRESSURE DISTRIBUTION IN OPERATION OF MULTISTRATUM PETROLEUM DEPOSITS

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 137-141, 1966

ABSTRACT: A study is made of the pressure distributions in the two strata of a circular deposit (radius R_+) extracted by different well systems. It is assumed that the two strata are separated by a stratum with much inferior collector properties.

The problem is solved via a finite Hankel transformation on the basis of a continuous distribution of sinks [1]. It is assumed that the flow may be averaged over the height [2]. The usual symbols [3] for the elastic state are used.

We assume that in each of the two productive strata (radius R) has a circular oil pool (radius RA) concentric with the boundary, the oil being extracted by sinks continuously distributed with constant densities q_1 and q_2.

The pressure distributions $p_1(r, t)$ and $p_2(r, t)$ in these two strata then satisfy the following system of differential equations, which are readily derived from the equations of continuity and motion:

$$\frac{\partial^2 p_1}{\partial r^2}+\frac{1}{r}\frac{\partial p_1}{\partial r}-\Lambda_1(p_1-p_2)=\frac{1}{\varkappa_1}\frac{\partial p_1}{\partial t}+\frac{\mu}{k_1 b_1}q_1(r),$$
$$\frac{\partial^2 p_2}{\partial r^2}+\frac{1}{r}\frac{\partial p_2}{\partial r}-\Lambda_2(p_2-p_1)=\frac{1}{\varkappa_2}\frac{\partial p_2}{\partial t}+\frac{\mu}{k_2 b_2}q_2(r). \quad (1)$$

Here

$$\Lambda_1=\frac{k_3}{b_3 b_1 k_1},\qquad \Lambda_2=\frac{k_3}{b_3 b_2 k_2},$$

$$q_{1,2}(r)=\begin{cases} q_{1,2} & (0\leqslant r<R_-)\\ 0 & (R_-<r\leqslant R_+)\end{cases} \quad (2)$$

The volume flow rates Q_1 and Q_2 in the two strata are

$$Q_1=\pi R_-^2 q_1,\qquad Q_2=\pi R_-^2 q_2. \quad (3)$$

The pressure at t = 0 is everywhere p_0. The pressure at the boundary remains at this value.

To simplify the symbolism we use the dimensionless quantities

$$r^\times=\frac{r}{R_+},\quad R=\frac{R_-}{R_+},\quad \tau=\frac{\varkappa_1 t}{R_+^2},$$

$$\varkappa=\frac{\varkappa_1}{\varkappa_2},\quad \lambda_{1,2}=\Lambda_{1,2}\cdot R_+^2,\quad Q=\frac{Q_2}{Q_1};$$

$$q_1^\times=\frac{2\pi R_+^2}{Q_1}q_1,\quad q_2^\times=\frac{2\pi b_1 k_1 R_+^2}{b_2 k_2 Q_1}q_2,$$
$$p_{1,2}^\times=\frac{2\pi b_1 k_1}{Q_1\mu}(p_0-p_{1,2}),\quad \beta=\frac{k_1 b_1}{k_2 b_2}. \quad (4)$$

The subsequent exposition is in terms of dimensionless quantities, and the x is everywhere omitted.

The problem reduces to solution of a system of differential equations in partial derivatives,

$$\frac{\partial^2 p_1}{\partial r^2}+\frac{1}{r}\frac{\partial p_1}{\partial r}-\lambda_1(p_1-p_2)=\frac{\partial p_1}{\partial \tau}-q_1(r)$$
$$\frac{\partial^2 p_2}{\partial r^2}+\frac{1}{r}\frac{\partial p_2}{\partial r}-\lambda_2(p_2-p_1)=\varkappa\frac{\partial p_2}{\partial \tau}-q_2(r)$$

$$q_{1,2}(r)=\begin{cases} q_{1,2} & (0\leqslant r\leqslant R),\\ 0 & (R<r\leqslant 1),\end{cases} \quad (5)$$

subject to the following initial and boundary conditions:

$$p_{1,2}=0 \quad \text{for } 0\leqslant r\leqslant 1,\quad \tau=0, \quad (6)$$

$$p_{1,2}=0 \quad \text{for } r=1,\quad \tau\geqslant 0, \quad (7)$$

and to the condition at the center of the strata,

$$r\frac{\partial p_{1,2}}{\partial r}\Big|_{r=0}=0. \quad (8)$$

We apply to (5) and (6) the finite Hankel transformation defined by

$$F(s)=\int_0^1 f(r)\, rJ_0(sr)\,dr, \quad (9)$$

where the s are the positive roots a_n (n = 1, 2, ...) of

$$J_0(a_n)=0. \quad (10)$$

Here $J_\nu(z)$ is a cylindrical function of the first kind of order ν.

As a result, we have to solve the following system of ordinary differential equations in order to get the Hankel transforms $P_{1,2}(s, \tau)$ of $p_{1,2}(r, t)$:

$$\frac{dP_1}{d\tau}=-a_n^2P_1-\lambda_1(P_1-P_2)+q_1\frac{R}{a_n}J_1(Ra_n),$$
$$\varkappa\frac{dP_2}{d\tau}=-a_n^2P_2-\lambda_2(P_2-P_1)+q_2\frac{R}{a_n}J_1(Ra_n) \quad (11)$$

subject to the initial condition

$$P_1=P_2=0 \quad \text{for } \tau=0. \quad (12)$$

The solution to (11) may be written as

$$P_i(a_n,\tau)=A_i(a_n)+B_i(a_n)\exp\eta_1(a_n)\tau+$$
$$+C_i(a_n)\exp\eta_2(a_n)\tau \quad (i=1, 2). \quad (13)$$

Here

$$A_1(a_n)=\frac{R}{a_n^3}J_1(Ra_n)\frac{(a_n^2+\lambda_2)q_1+\lambda_1 q_2}{a_n^2+\lambda_1+\lambda_2},$$

$$A_2(a_n)=\frac{R}{a_n^3}J_1(Ra_n)\frac{(a_n^2+\lambda_2)q_2+\lambda_2 q_1}{a_n^2+\lambda_1+\lambda_2},$$

$$B_1(a_n)=q_1\frac{R}{a_n}J_1(Ra_n)\frac{\eta_1(a_n)+(a_n^2+\lambda_2)/\varkappa+\lambda_1 q_2/q_1\varkappa}{\eta_1(a_n)[\eta_1(a_n)-\eta_2(a_n)]},$$

$$B_2(a_n)=\frac{q_2R}{\varkappa a_n}J_1(Ra_n)\frac{\eta_1(a_n)+a_n^2+\lambda_1+\lambda_2 q_1/q_2}{\eta_1(a_n)[\eta_1(a_n)-\eta_2(a_n)]},$$

$$C_1(a_n)=q_1\frac{R}{a_n}J_1(Ra_n)\frac{\eta_2(a_n)+(a_n^2+\lambda_2)/\varkappa+\lambda_1 q_2/q_1\varkappa}{\eta_2(a_n)[\eta_2(a_n)-\eta_1(a_n)]},$$

$$C_2(a_n)=\frac{q_2R}{\varkappa a_n}J_1(Ra_n)\frac{\eta_2(a_n)+a_n^2+\lambda_1+\lambda_2 q_1/q_2}{\eta_2(a_n)[\eta_2(a_n)-\eta_1(a_n)]},$$

$$\eta_1(a_n)=-\frac{1}{2}\Big[a_n^2\Big(1+\frac{1}{\varkappa}\Big)+\lambda_1+\frac{\lambda_2}{\varkappa}-$$
$$-\Big(\Big(a_n^2\frac{\varkappa-1}{\varkappa}+\lambda_1-\frac{\lambda_2}{\varkappa}\Big)^2+4\frac{\lambda_1\lambda_2}{\varkappa}\Big)^{1/2}\Big],$$

$$\eta_2(a_n)=-\frac{1}{2}\Big[a_n^2\Big(1+\frac{1}{\varkappa}\Big)+\lambda_1+\frac{\lambda_2}{\varkappa}+$$
$$+\Big(\Big(a_n^2\frac{\varkappa-1}{\varkappa}+\lambda_1-\frac{\lambda_2}{\varkappa}\Big)^2+4\frac{\lambda_1\lambda_2}{\varkappa}\Big)^{1/2}\Big]. \quad (14)$$

We convert from the transforms to the original via the following formula:

$$f(r)=2\sum_{n=1}^{\infty}\frac{J_0(a_n r)}{J_1^2(a_n)}F(a_n), \qquad (15)$$

which gives for the dimensionless pressures

$$p_i(r,\tau)=2\sum_{n=1}^{\infty}A_i(a_n)\frac{J_0(a_n r)}{J_1^2(a_n)}+ \\ +2\sum_{n=1}^{\infty}[B_i(a_n)\exp\eta_1(a_n)\tau+C_i(a_n)\exp\eta_2(a_n)\tau]\frac{J_0(a_n r)}{J_1^2(a_n)}. \qquad (16)$$

Now η_1 and η_2 are negative for any values of the parameters of stratum and fluid, and also for all $a_n (n = 1, 2, \ldots)$ while their moduli as n increases become larger than a_n^2, so the second series on the right in (16) converge rapidly for τ not close to zero. The first series on the right in (16) do not converge so rapidly though.

The distribution tends to a stationary one as τ tends to infinity in (16), and we have

$$p_i(r)=2\sum_{n=1}^{\infty}A_i(a_n)\frac{J_0(a_n r)}{J_1^2(a_n)} \qquad (i=1,2). \qquad (17)$$

On the other hand, these functions must satisfy (5) if we put $\partial p_1/\partial\tau \equiv \partial p_2/\partial\tau \equiv 0$. We discard the time derivatives in (5) and solve the resulting system of ordinary differential equations subject to (7), which gives for $p_1(r)$ and $p_2(r)$

$$0\leqslant r\leqslant R)$$

$$p_i(r)=-\frac{q_1\lambda_2+q_2\lambda_1}{\lambda_1+\lambda_2}\left\{\frac{R^2}{2}\ln R+\frac{r^2-R^2}{4}\mp\frac{\lambda_i(q_1-q_2)}{\omega^2(q_1\lambda_2+q_2\lambda_1)}\pm\right. \\ \left.\pm\frac{\lambda_i R(q_1-q_2)}{\omega(q_1\lambda_2+q_2\lambda_1)}[K_1(R\omega)I_0(\omega)+I_1(R\omega)K_0(\omega)]\frac{I_0(r\omega)}{I_0(\omega)}\right\},$$

$$(0\leqslant r\leqslant R)$$

$$p_i(r)=-\frac{q_1\lambda_2+q_2\lambda_1}{\lambda_1+\lambda_2}\left\{\frac{R^2}{2}\ln r\mp\frac{\lambda_i R(q_1-q_2)}{\omega(q_1\lambda_2+q_2\lambda_1)}\times\right. \\ \left.\times[I_0(\omega)K_0(r\omega)-K_0(\omega)I_0(r\omega)]\frac{I_1(R\omega)}{I_0(\omega)}\right\}$$

$$(R\leqslant r\leqslant 1), \qquad (18)$$

in which $\omega = \sqrt{\lambda_1+\lambda_2}$, and $I_0(z)$, $I_1(z)$, $K_0(z)$, $K_1(z)$ are modified cylindrical functions of the first and second kinds of the corresponding orders.

To obtain the solutions of (18) we have used the general solutions [2] of the homogeneous system corresponding to (5) and the method of constant variation.

From (16)-(18) we write the solution to the problem as

$$p_i(r,\tau)=p_i(r)+2\sum_{n=1}^{\infty}[B_i(a_n)\exp\eta_1(a_n)\tau+ \\ +C_i(a_n)\exp\eta_2(a_n)\tau]\frac{J_0(a_n r)}{J_1^2(a_n)}, \qquad (19)$$

in which $B_i(a_n)$, $C_i(a_n)$ and $\eta_{1,2}(a_n)$ are defined by (14) and $p_i(r)$ by (18).

This problem has been solved in the most general formulation, where all parameters of the strata and interlayer are different. From the universal solution of (19) we readily get various particular cases, e.g., $\varkappa_1=\varkappa_2$, $\Lambda_1=\Lambda_2$, or $q_1=q_2$. The solutions then become much simpler.

Here we consider some particular cases that are not so obvious.

1. We let R_- tend to zero and the density of sinks to infinity in such a way that the flow rates remain constant at Q_1 and Q_2.

For this purpose

$$q_1=\frac{2}{R^2}, \qquad q_2=\frac{2}{R^2}\beta Q.$$

Then for $R\to 0$ the last terms on the right in (1) and (5) become zero, and conditions (8) become

$$r\left.\frac{\partial p_1}{\partial r}\right|_{r\to 0}=-1, \qquad r\left.\frac{\partial p_2}{\partial r}\right|_{r\to 0}=-\beta Q. \qquad (20)$$

Further, the last terms on the right in (11) are replaced respectively by 1 and βQ.

Hence the entire solution is altered only as to symbols, and we get

$$p_i^*(r,\tau)=p_i^*(r)+2\sum_{n=1}^{\infty}[B_i^*(a_n)\exp\eta_1(a_n)\tau+ \\ +C_i^*(a_n)\exp\eta_2(a_n)\tau]\frac{J_0(a_n r)}{J_1^2(a_n)} \\ (i=1,2). \qquad (21)$$

Here

$$p_i^*(r)=-\frac{\lambda_2+\lambda_1\beta Q}{\lambda_1+\lambda_2}\ln r\pm\lambda_i\frac{1-\beta Q}{\lambda_1+\lambda_2}\times \\ \times\frac{I_0(\omega)K_0(r\omega)-K_0(\omega)I_0(r\omega)}{I_0(\omega)},$$

$$B_1^*(a_n)=\frac{\eta_1(a_n)+(a_n^2+\lambda_2)/\varkappa+\lambda_1\beta Q/\varkappa}{\eta_1(a_n)[\eta_1(a_n)-\eta_2(a_n)]},$$

$$B_2^*(a_n)=\frac{\beta Q[\eta_1(a_n)+a_n^2+\lambda_1+\lambda_2/Q]}{\varkappa\eta_1(a_n)[\eta_1(a_n)-\eta_2(a_n)]},$$

$$C_1^*(a_n)=\frac{\eta_2(a_n)+(a_n^2+\lambda_2)/\varkappa+\lambda_1\beta Q/\varkappa}{\eta_2(a_n)[\eta_2(a_n)-\eta_1(a_n)]},$$

$$C_2^*(a_n)=\frac{\beta Q[\eta_2(a_n)+a_n^2+\lambda_1+\lambda_2/Q]}{\varkappa\eta_2(a_n)[\eta_2(a_n)-\eta_1(a_n)]}. \qquad (22)$$

Formulas (21) and (22) allow us to find the distribution of the dimensionless pressure drop in both strata when these are exploited by boreholes of infinitely small diameter located at the center.

2. We put first $R=R^0$ and then $R=R^0+\Delta R$ in (19) and subtract the first result from the second; this gives us formulas for the pressure reduction due to rings of sinks continuously distributed, the internal dimensionless radii of the rings being R^0 and the dimensionless width ΔR. Next we let ΔR tend to zero, with the density of sinks tending to infinity in such a way that the flow rates from the rings remain constant at Q_1 and Q_2; this gives us formulas for exploitation at rates Q_1 and Q_2 in the form

$$p_i^\circ(r,\tau)=p_i^\circ(r)+2\sum_{n=1}^{\infty}[B_i^\circ(a_n)\exp\eta_1(a_n)\tau+ \\ +C_i^\circ(a_n)\exp\eta_2(a_n)\tau]\frac{J_0(a_n r)}{J_1^2(a_n)}, \qquad (23)$$

$$p_i^\circ(r)=-\frac{\lambda_2+Q\beta\lambda_1}{\lambda_1+\lambda_2}\ln R^\circ\mp\lambda_i\frac{1-\beta Q}{\lambda_1+\lambda_2}[I_0(R^\circ\omega)K_0(\omega)- \\ -K_0(R^\circ\omega)I_0(\omega)]\frac{I_0(r\omega)}{I_0(\omega)},$$

$$(0\leqslant r\leqslant R^\circ) \qquad (i=1,2), \qquad (24)$$

$$B_1^\circ(a_n)=J_0(R^\circ a_n)\frac{\eta_1(a_n)+(a_n^2+\lambda_2)/\varkappa+\lambda_1\beta Q}{\eta_1(a_n)[\eta_1(a_n)-\eta_2(a_n)]},$$

$$B_2^\circ(a_n)=\frac{Q\beta}{\varkappa}J_0(R^\circ a_n)\frac{\eta_1(a_n)+a_n^2+\lambda_1+\lambda_1/Q}{\eta_1(a_n)[\eta_1(a_n)-\eta_2(a_n)]},$$

$$C_1^\circ(a_n)=J_0(R^\circ a_n)\frac{\eta_2(a_n)+(a_n^2+\lambda_2)/\varkappa+Q\beta\lambda_1/\varkappa}{\eta_2(a_n)[\eta_2(a_n)-\eta_1(a_n)]},$$

$$C_2^\circ(a_n)=\frac{Q\beta}{\varkappa}J_0(R^\circ a_n)\frac{\eta_2(a_n)+a_n^2+\lambda_1+\lambda_1/Q}{\eta_2(a_n)[\eta_2(a_n)-\eta_1(a_n)]}. \qquad (25)$$

For r in the range $R^0 \le r \le 1$ we interchange r and R^0 in (24).

3. Solutions (19) and (23) may be used if the radii of the circular regions of continuously distributed sinks are different in the two strata. For this purpose we first put $Q_2 = 0$ and $R = R_1$ or $R^0 = R_1$, and then $Q_1 = 0$ with $R = R_2$, the results being then added.

Leakage through top and bottom (or full flows) can occur if there is any great difference between the pressures, and this volume of liquid cannot be neglected.

Problems of leakage through low-permeability strata have . ..ne of some interest.*

The formulas derived here enable us to find the flow between strata as follows:

$$Q_0 = \frac{k_3}{\mu b_3} 2\pi \int_0^{R_+} (p_1 - p_2)\, r dr,$$

or in the dimensionless terms of (4),

$$Q_+ = \frac{Q_0}{Q_1} = \lambda_1 \int_0^1 (p_2 - p_1)\, r dr. \qquad (26)$$

We substitute the p_1 and p_2 of (19), (21), and (23) into (20) and integrate to get the flow from the second stratum into the first when the strata are exploited by:

a) sinks continuously distributed over the area

$$Q_+ = \lambda_1 \frac{q_2 - q_1}{\omega^2}\left[\frac{R^2}{2} - \frac{R I_1(R\omega)}{\omega I_0(\omega)}\right] + 2\lambda_1 \sum_{n=1}^{\infty}{}' \times$$
$$\times \{[B_2(a_n) - B_1(a_n)] \exp \eta_1(a_n)\tau + [C_2(a_n) -$$
$$- C_1(a_n)] \exp \eta_2(a_n)\tau\} \frac{1}{a_n J_1(a_n)}; \qquad (27)$$

b) point sinks at the center

$$Q_+{}^* = \lambda_1 \frac{1-\beta Q}{\omega^2}\left[\frac{1 - I_0(\omega)}{I_0(\omega)}\right] + 2\lambda_1 \sum_{n=1}^{\infty} \times$$
$$\times \{[B_2{}^*(a_n) - B_1{}^*(a_n)] \exp \eta_1(a_n)\tau +$$
$$+ [C_2{}^*(a_n) - C_1{}^*(a_n)] \exp \eta_2(a_n)\tau\} \frac{1}{a_n J_1(a_n)}; \qquad (28)$$

c) circular galleries concentric with the boundaries

$$Q_+{}^\circ = \lambda_1 \frac{1-\beta Q}{\omega^2}\left[\frac{I_0(R^\circ\omega)}{I_0(\omega)} - 1\right] + 2\lambda_1 \sum_{n=1}^{\infty} \times$$
$$\times \{[B_2{}^\circ(a_n) - B_1{}^\circ(a_n)] \exp \eta_1(a_n)\tau +$$
$$+ [C_2{}^\circ(a_n) - C_1{}^\circ(a_n)] \exp \eta_2(a_n)\tau\} \frac{1}{a_n J_1(a_n)}. \qquad (29)$$

All the formulas derived by rigorous hydrodynamic methods are adequate for practical calculations and allow one to deduce the pressure in both strata and the flow between them.

We are indebted to V. N. Shchelkachev for proposing this topic and for valuable comments on the work.

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7 May 1965 Moscow

*See [2] for a brief survey of these problems and of methods of solution.